Minimization Problems Generated by a Closed Convex Set and a Smooth Curve in the Plane

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Abstract

Let $H \subset \mathbb{R}^2$ be a closed convex set with nonempty interior $int(H)$ and γ be a smooth curve in the same space. For a point $A \in \gamma$ denote by $n_\gamma(A)$ the normal to γ at the point A. Put $D = \{A \in \gamma : n_\gamma(A) \cap int(H) \neq \emptyset\}$ and consider in D the function $f(A) = |A_1A_2|$ where $|A_1A_2|$ is the Euclidean distance between the end-points A_1 and A_2 of the non-degenerated segment $[A_1A_2] = n_\gamma(A) \cap H$.

Theorem 1 *Let γ be a C^2-smooth curve with non-vanishing curvature. Let A^* be a local minimum in D for the function $f(A)$ with $n_\gamma(A^*) \cap H = [A_1^*A_2^*]$. Suppose also that the centre of curvature $C_\gamma(A^*)$ of γ at A^* does not belong to the boundary ∂H of H. Then*

(i) the boundary ∂H is smooth at the points A_1^* and A_2^*. I.e. at each of the points A_1^* and A_2^* there is only one supporting line for the set H, and, consequently, the normals $n_{\partial H}(A_i^*)$, $i = 1, 2$, to ∂H are well defined ("smoothness property").

(ii) there exists a point P belonging to the following three normals: $n_{\partial H}(A_i^*)$, $i = 1, 2$, and the normal $n(C_\gamma(A^*))$ to the line $A_1^*A_2^*$ at the point $C_\gamma(A^*)$ ('‘three concurrent normals property").

The particular case of this minimization problem, when γ is a circle with center O, reduces to finding, among all the lines through O, the lines that cut from H locally shortest segments.

Instead of considering the normals to the curve γ, one could take the tangents $t_\gamma(A)$ to γ, where $A \in \gamma$, and look for those tangents that cut from H locally shortest non-degenerated segments. Both the "smoothness property" and the "three concurrent normals property" remain valid.

These results have been obtained in co-authorship with Oleg Mushkarov and Nikolai Nikolov.